Approximating Likelihoods for Spatial Extremes with Deep Learning

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Background

Motivation

Figure 1: Sample 0.9 quantile of log annual streamflow maxima at 489 locations.
Source: USGS Hydro Climatic Data Network (HCDN).

- Extremal streamflow is a key measure of flood risk
- Quantifying how the probability and magnitude of extreme flooding events are changing is key to mitigating their impacts under changing climate

Max-stable processes for spatial extremes

- Gaussian processes (GP) are inadequate for modeling extremes
- Max-stable processes (MSP) are a natural model for block maxima, $d_{\mathfrak{b}}\omega'' \bar{a}^{\cdot} \bar{a}_{\mathfrak{w}}^{\mathfrak{w}}$:
  - Intractable likelihood for even moderately large problems
  - Restrictive in the class of dependence types they can incorporate
- **Approximation** - Composite Likelihood[1]
  - Inefficient, finite sample bias, computational challenges for large $\mathfrak{h}$
- **Approximation** - Vecchia approximation
  - Simplifies likelihoods for spatial processes including MSPs[2]

[1] Padoan $\bar{a}^{-}$ $h\mathfrak{y}^{i}$ (2010)
[2] Huser $\bar{a}^{-}$ $h\mathfrak{y}^{i}$ (2022)

Objectives

- For large spatial extremes datasets, we want:
  - Expressive and flexible spatial processes
  - Computational strategies for intractable likelihoods
- **Our approach** - Process mixture model specified as a convex combination of a GP and an MSP
- Vecchia approximation simplifies likelihood as a product of univariate (intractable) PDFs
- Deep learning to approximate the intractable PDFs

The Process Mixture Model

The process mixture model (PMM)

- Let $\tilde{U}(\mathbf{s})$ be the extreme observation at spatial location $\mathbf{s}$ with a generalized extreme value (GEV) distribution:

$$\tilde{U}(\mathbf{s}) \quad \text{GEV}\{ \quad (\mathbf{s}); \quad (\mathbf{s}); \quad (\mathbf{s})\}$$

- $\tilde{U}(\mathbf{s}) \quad \{_{\mathbf{s}}, L(\mathbf{s}) = \{_{\mathbf{s}}(\tilde{U}(\mathbf{s}))$, and express the joint likelihood as

$$\ddot{\imath}(\ _1; \ldots; \ _{\text{ﺩ}}; \ _1; \ _2) = \ddot{\imath}(\ _1; \ldots; \ _{\text{ﺩ}}; \ _2) \prod_{\text{ﺩ}=1}^{\text{ﺩ}} \frac{\breve{a}\{_{\mathbf{s}}(\ _{\text{ﺩ}}; \ _1)}{\breve{a}\ _{\text{ﺩ}}} \ ; \qquad (1)$$

where $\ _{\text{ﺩ}} \quad (\mathbf{s}_{\text{ﺩ}})$ and $\ _{\text{ﺩ}} = \{_{\mathbf{s}}(\ _{\text{ﺩ}}; \ _1)$

- Take $L(\mathbf{s}) = \sim\{\hat{S}(\mathbf{s})\}$ to get spatial dependence model on $L(\mathbf{s})$

$$\hat{S}(\mathbf{s}) = \quad \hat{O}_{\hat{G}}\{\hat{G}(\mathbf{s})\} + (1 \quad) \ \hat{O}_{\check{S}}\{\check{S}(\mathbf{s})\} \qquad (2)$$

Spatial dependence in the PMM

- Take $L(\mathbf{s}) = \sim f\hat{S}(\mathbf{s})g$ to get spatial dependence model on $L(\mathbf{s})$

$$\hat{S}(\mathbf{s}) = \quad \hat{O}_{\hat{G}} f\hat{G}(\mathbf{s})g + (1 \quad) \quad \hat{O}_{\check{S}} f\check{S}(\mathbf{s})g$$

- $\hat{G}(\mathbf{s})$ is an MSP, $\check{S}(\mathbf{s})$ is a GP; $\quad 2\ [0;1]$
- Conditional exceedance probability defined as:

$$_{\cdot}(\mathbf{s}_1;\mathbf{s}_2) := \text{Prob} f L(\mathbf{s}_1) > {}_{\cdot} j L(\mathbf{s}_2) > {}_{\cdot} g$$

- $\quad (\mathbf{s}_1;\mathbf{s}_2) = \lim_{\cdot\ !\ 1} \quad {}_{\cdot}(\mathbf{s}_1;\mathbf{s}_2) > 0$ iff $\quad > 0{:}5 \quad =)$ asymptotic dependence
- $\hat{O}_{\hat{G}} f\hat{G}(\mathbf{s})g; \hat{O}_{\check{S}} f\check{S}(\mathbf{s})g$ $^{\ddot{\ddot{0}}\check{a}}$ Exponential(1)
- Process mixture $\hat{S}(\mathbf{s})$ - hypoexponential distribution marginally
- Generalization of $^{a}{}_{\cdot}$ ’$\bar{a}^{w}_{w}$ $hu\hbar\check{a}$ $\check{S}hu\check{a}$ ’$''$ $\omega^{w}_{w}{}^{-}\dot{d}$ ㄴ╖ ╔╝ ㅁ.

Deep Learning Vecchia approximation for the PMM

- Joint likelihood:

$$\tilde{\ell}(\tilde{\ }_1;\ldots;\tilde{\ }_n;\ _1;\ _2) = \tilde{\ell}_{\tilde{\ }}(\tilde{\ }_1;\ldots;\tilde{\ }_n;\ _2)\prod_{i=1}^{n}\frac{\tilde{a}\{S(\tilde{\ }_i;\ _1)}{\tilde{a}\tilde{\ }_i}$$

- Approximate the first term of the likelihood as[3]

$$\tilde{\ell}_{\tilde{\ }}(\tilde{\ }_1;\ldots;\tilde{\ }_n;\ _2) = \prod_{i=1}^{n}\tilde{\ell}(\tilde{\ }_i|\ _2;\tilde{\ }_1;\ldots;\tilde{\ }_{i-1})\prod_{i=1}^{n}\tilde{\ell}_i(\tilde{\ }_i|\ _2;\tilde{\ }_{(i)}); \quad (3)$$

for $\tilde{\ }_{(i)} = \{\tilde{\ }_j; j \in N_i\}$ and neighboring set $N_i \subset \{1;\ldots;i-1\}$

- $\tilde{\ }_{(i)}$: Vecchia neighboring set.

[3] Vecchia (1988), Stein $\bar{a}^{-}$ hwy^{i} (2004), Datta $\bar{a}^{-}$ hwy^{i} (2016), Katzfuss and Guinness (2021)

Deep Learning Vecchia approximation for the PMM

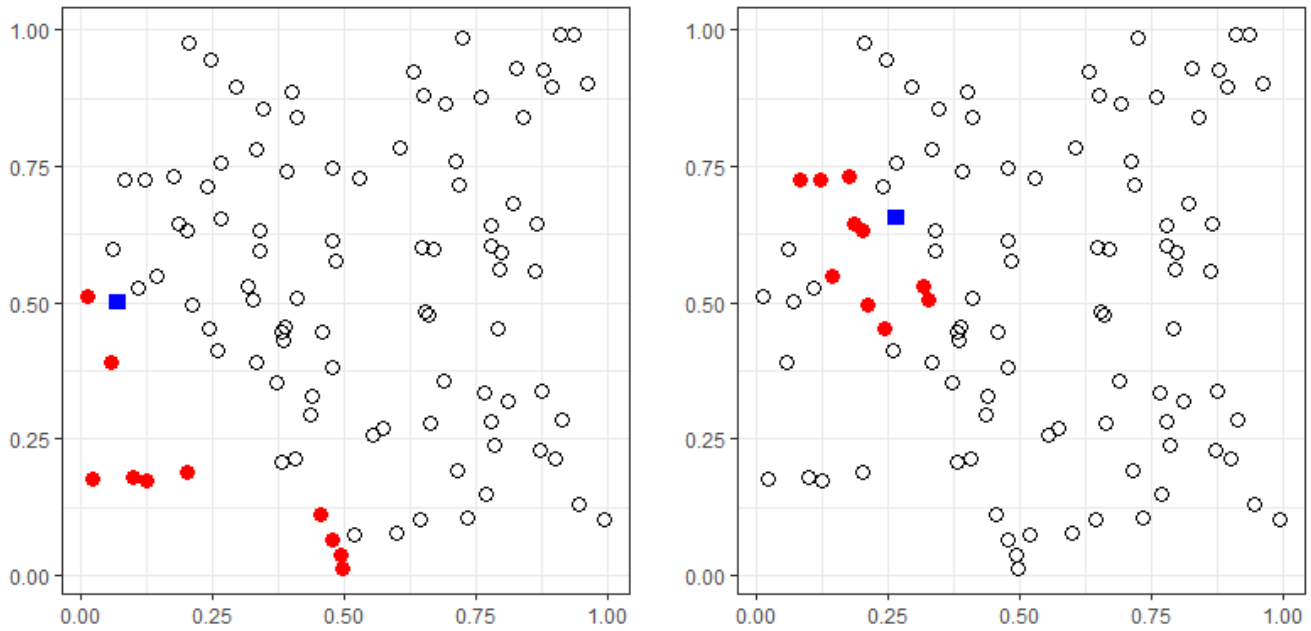


Figure 2: Vecchia neighboring sets when locations are ordered by distance from origin

- The Vecchia neighboring set has up to 10 locations in this example
- No analytical form for $\grave{\imath}_{\hat{\imath}}(\breve{\imath}_{\hat{\imath}}j\ \ _2; \breve{\imath}_{(\hat{\imath})})$

Deep Learning Vecchia approximation for the PMM

- Model $\ddot{\imath}_{\mathbb{j}}(\breve{\imath}_{\mathbb{j}} j\ _2; \breve{\imath}_{(\mathbb{j})})$ using semi parametric quantile regression (SPQR)[4] as:

$$\ddot{\imath}(\breve{\imath}_{\mathbb{j}} j \mathbf{x}_{\mathbb{j}}; W_{\mathbb{j}}) = \sum_{j=1}^{\acute{\mathbb{X}}} {}_j(\mathbf{x}_{\mathbb{j}}; W_{\mathbb{j}}) \quad @_j(\breve{\imath}_{\mathbb{j}}) \tag{4}$$

- M-spline basis functions $@_j(\breve{\imath})$ 0: satisfy $\int^{\mathrm{R}} @_j(\breve{\imath})\breve{a}\breve{\imath} = 1$ for all k
- Probability weights ${}_j(\mathbf{x}_{\mathbb{j}}; W)$: softmax outputs from a feed-forward neural network (FFNN)
- Can approximate conditional densities smooth in its arguments[5]

[4] Xu and Reich (2021)
[5] Chui $\bar{a}^{-}$ $h\text{y}^{i}$ (1980), Hornik $\bar{a}^{-}$ $h\text{y}^{i}$ (1989)

SPQR implementation

- Each $\grave{\imath}(\breve{\ }_{\grave{\jmath}} j \mathbf{x}_{\grave{\jmath}}; W_{\grave{\jmath}})$ is modeled using its own FFNN; $\mathbf{x}_{\grave{\jmath}} := (\ \ _2; \breve{\ }_{(\grave{\jmath})})$
- FFNN weights $W_{\grave{\jmath}}$ for location $\grave{\jmath}$ estimated using synthetic data generated using plausible parameter values
- Parameter estimation carried out afterwards using MCMC

Numerical Results

Simulation Study - Process Mixture Model

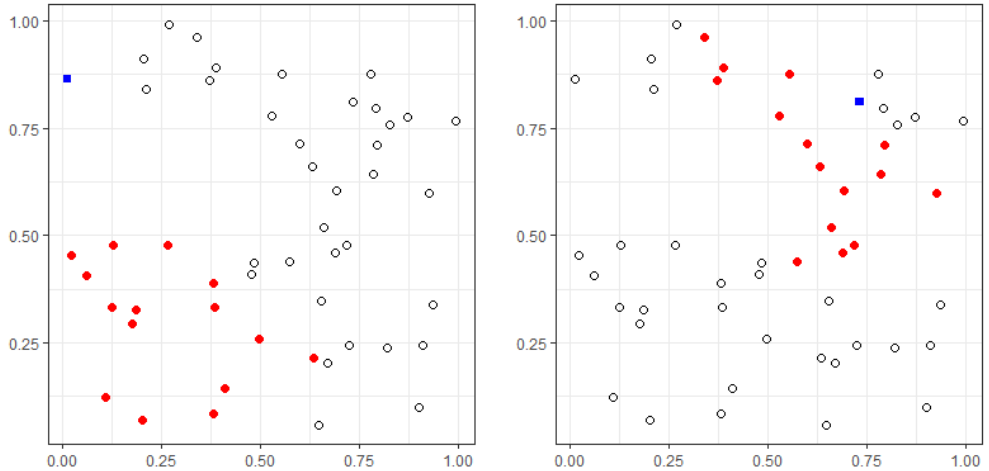
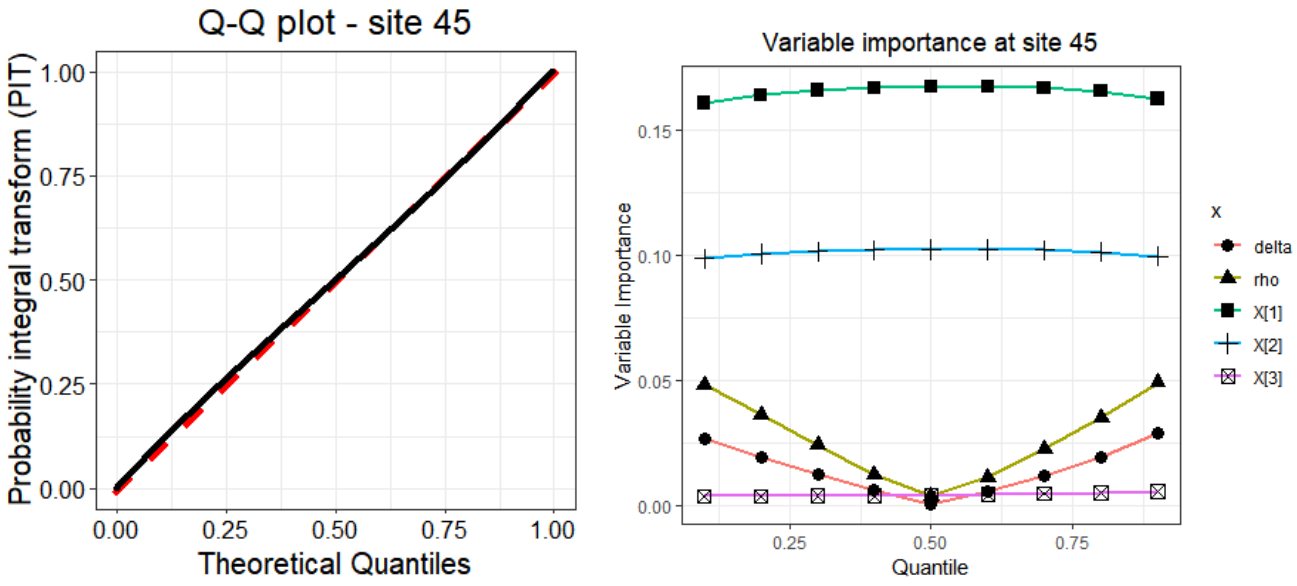


Figure 3: Locations used in the EVP simulation studies: 50 locations, and nearest neighbor assignments for locations 16 (left) and 45 (right).

- Common smoothness parameter $\hat{G} = \check{S} = \quad = 1$
- Range $= \check{S}, \hat{G} = 0{:}19$
- Range chosen such that distance at which GP correlation reaches 0.05 = distance at which $\cdot(\mathbf{s}_1; \mathbf{s}_2)$ for MSP is 0.05, where

$$\cdot(\mathbf{s}_1; \mathbf{s}_2) := \text{Prob}\{L(\mathbf{s}_1) > \cdot\, jL(\mathbf{s}_2) > \cdot\, g$$

SPQR model fit diagnostics - PMM



(a) Q-Q plots for checking goodness of fit.

(b) Variable importance of , , and nearest neighbors.

Figure 4: Model diagnostics for process mixture model: Q-Q plot and VI plot.

SPQR settings: 50 epochs, batch size 100, learning rate 0.001, 2 hidden layers (30, 15 neurons), 15 output knots, 10^6 obs.

SPQR model fit diagnostics

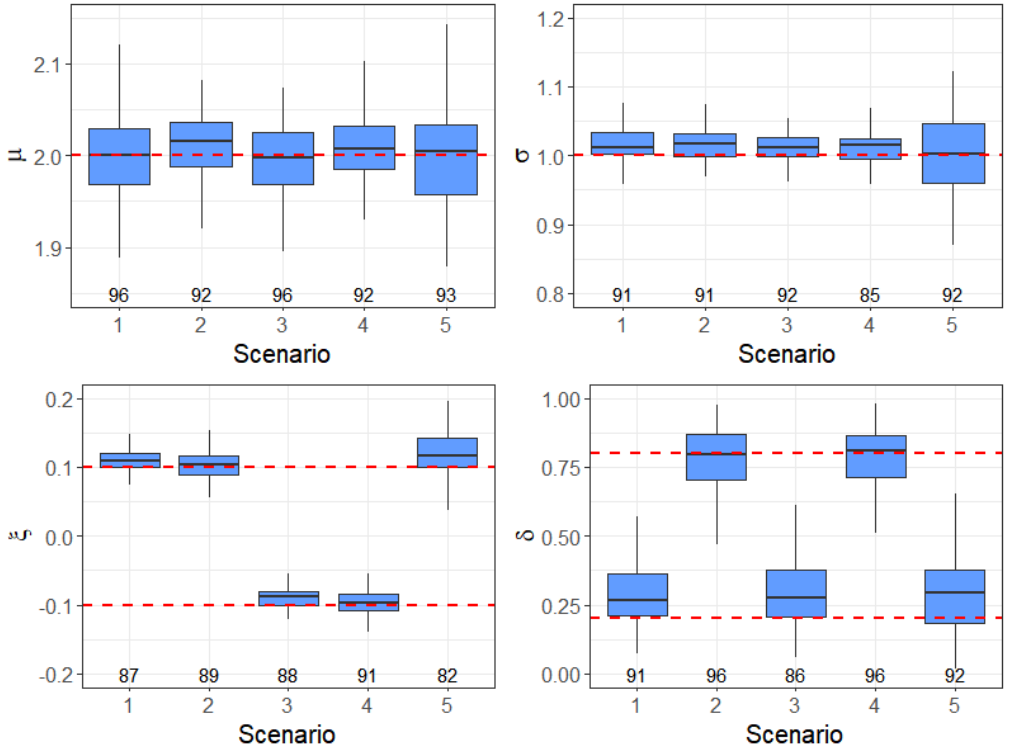


Figure 5: Sampling distribution of posterior means: Horizontal dashed lines are true values with empirical coverage of the 95% intervals at the bottom.

- Scenario 5: MCAR with probability $\grave{U} = 0{:}05$ and censored below the threshold $\dot{I} = \hat{t}c_{0.5}$ (over space and time)

Case Study: Extreme Streamflow

Case study: extreme streamflow data

Figure 6: Sample 0.9 quantile of log annual streamflow maxima $\tilde{U}$-(’) at 489 locations.

- **489 locations** across the US part of the USGS Hydro-Climatic Data Network (HCDN)
- **50 years** of complete data from 1972–2021 - annual streamflow maxima

Spatio-temporally varying coefficients model for the marginals

- $\tilde{U}_{\bar{}}(\mathbf{s})$: log annual maxima for year $\bar{}$, location $\mathbf{s}$
- GEV marginals with spatio-temporally varying coefficients (STVC):

$$\tilde{U}_{\bar{}}(\mathbf{s}) \quad \text{GEV}[\ {}_0(\mathbf{s}) + \ {}_1(\mathbf{s})t_{\bar{}}; \exp f\ \ (\mathbf{s})g;\ \ (\mathbf{s})]; \tag{5}$$

$t_{\bar{}} = (\text{year}_{\bar{}} \quad 1996{:}5)/10$ for $\text{year}_{\bar{}} = 1972 + \bar{} \quad 1$

- $t_{\bar{}}$ captures changes in the location due to changing climate
- $(\ {}_0(\mathbf{s});\ {}_1(\mathbf{s});\ \ (\mathbf{s});\ \ (\mathbf{s}))$ GPs with common range parameter
- FFNN architecture: 15 neighbors, 2 hidden layers (30, 20 neurons), 15 output knots, batch size 1000, learning rate 0.01, 50 epochs

Posterior estimates

Posterior means and SD of spatial parameter estimates:

- ^ : 0:47 (0:02); ^ : 1004 j g (80); ^ : 0:56 (0:07); ^ : 17907 j g (1806)
- Asymptotic independence regime with high probability

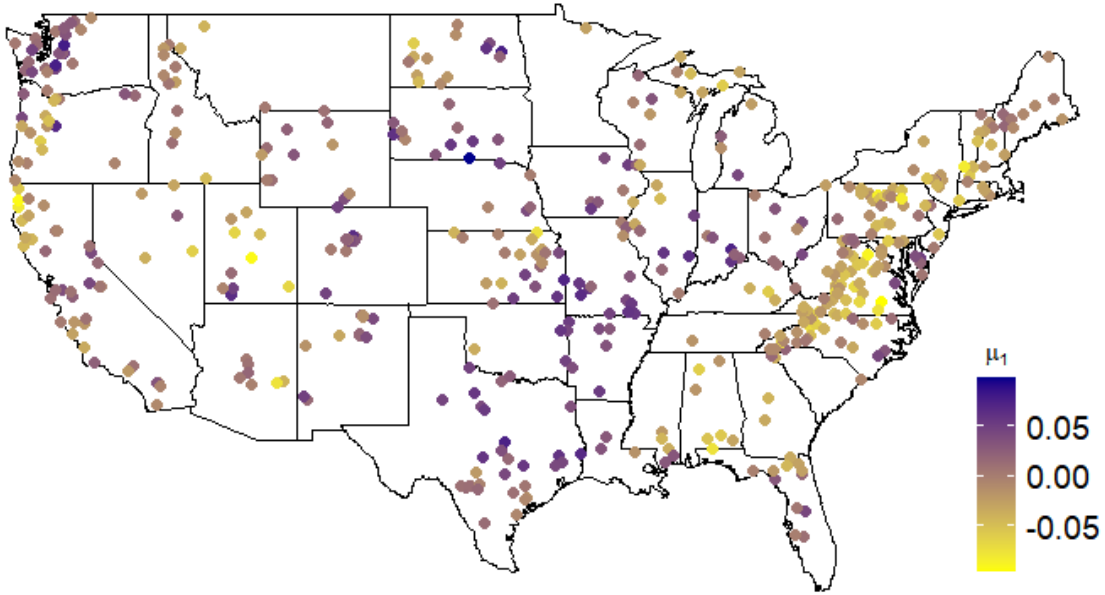


Figure 7: Posterior mean of $_1$(s) at 489 gauges for log annual streamflow maxima.

- Positive values of $_1$ indicate increasing streamflow maxima

Posterior estimates

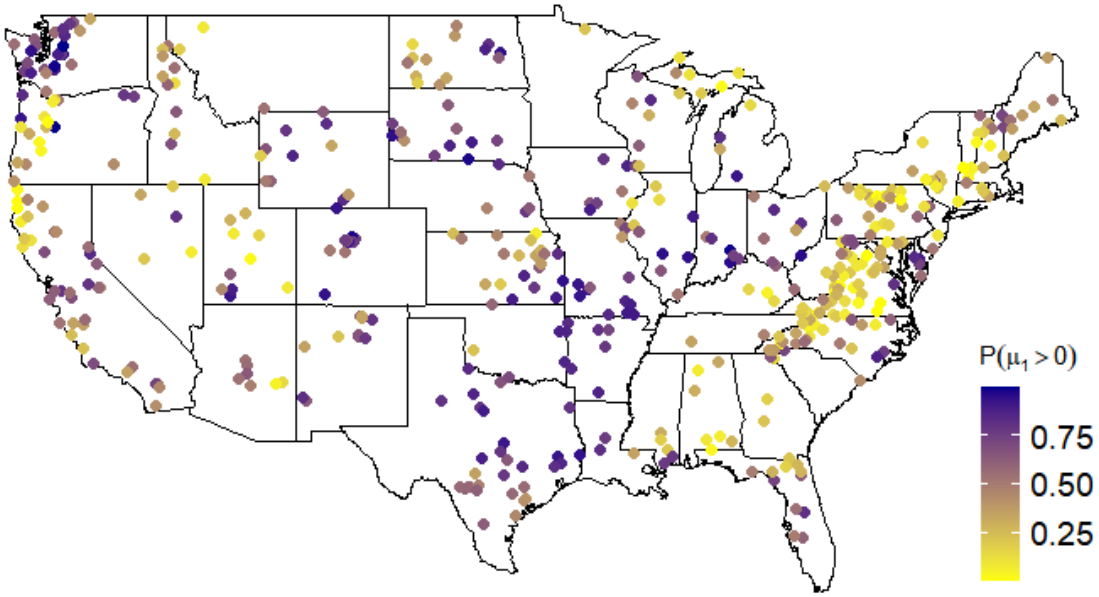


Figure 8: Estimates of $\check{e}^{W}_{W}[\ _1(\mathbf{s}) > 0]$ for the GEV location parameters.

- Higher values indicate stronger evidence of increased streamflow magnitude between 1972 and 2021
- Joint exceedances can be studied for clusters; e.g. in CO, posterior probability that 0.9 quantile has gone up is 0.975

Summary and Discussion

- Extreme value analysis of climate signals is of growing importance, but existing methods are often intractable
- The process mixture model identifies patterns of increasing streamflow due to changing climate within the US
- Flexible, tractable, parallelizable, can take advantage of GPU acceleration
- Main idea can be applied to virtually any spatial process